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AXISYMMETRIC TORSION PROBLEM FOR A TRUNCATED CONE CONTAINING A SPHERICAL CRACK

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Abstract: *The paper addresses the problem of torsion of a truncated cone with an internal spherical crack. The aim of the study is to determine the stress-strain state of the cone in the presence of a crack and applied axisymmetric tangential load on the surface. The problem is solved using the Legendre integral transformation, which reduces the original boundary problem to a one-dimensional discontinuous boundary problem. The solution is represented as a sum of continuous and discontinuous components obtained using Green's function. The unknown displacement jump at the crack is determined from the integral equation, solved by the method of orthogonal polynomials with the use of Chebyshev polynomials. As a result, the stress intensity factor near the crack is calculated, and a numerical analysis of the influence of geometric parameters on the stress state is carried out. The obtained results are important for assessing the strength and stability of structures with cracks and can be applied in engineering practice.*

Key words: *truncated cone, spherical crack, stress intensity factor, integral transformation, orthogonal polynomial method.*

Introduction.

Conical objects are frequently used in engineering practice and construction, which makes the study of their stress state a relevant problem both from a practical standpoint and in terms of advancing mathematical methods for solving such problems. Of particular interest are bodies that contain defects in the form of spherical or conical cracks, as these significantly affect the durability of corresponding structures. In this work, we consider the axisymmetric problem of torsion of a truncated cone (truncated along a spherical surface) containing a spherical crack. The conical surface is fixed, while an axisymmetric tangential stress applied to the spherical surface induces torsion.

There has been relatively little research dedicated to the stress state of elastic cones (solid, hollow, truncated along spherical surfaces) [1-5], which was largely due to the lack of suitable integral transformations. G.Ya. Popov proposed new integral transformations [6,7] with kernels in the form of associated Legendre functions, which allowed for solutions to a number of new problems for elastic cones, including those containing defects such as cracks. Using these integral transformations, several axisymmetric problems of elasticity theory for cones were solved in [8-10]. The presence of conical and spherical defects within a cone was addressed in [11-13]. Nonstationary problems for elastic cones, including those with cracks, were considered in [14-17].

1. Problem Statement

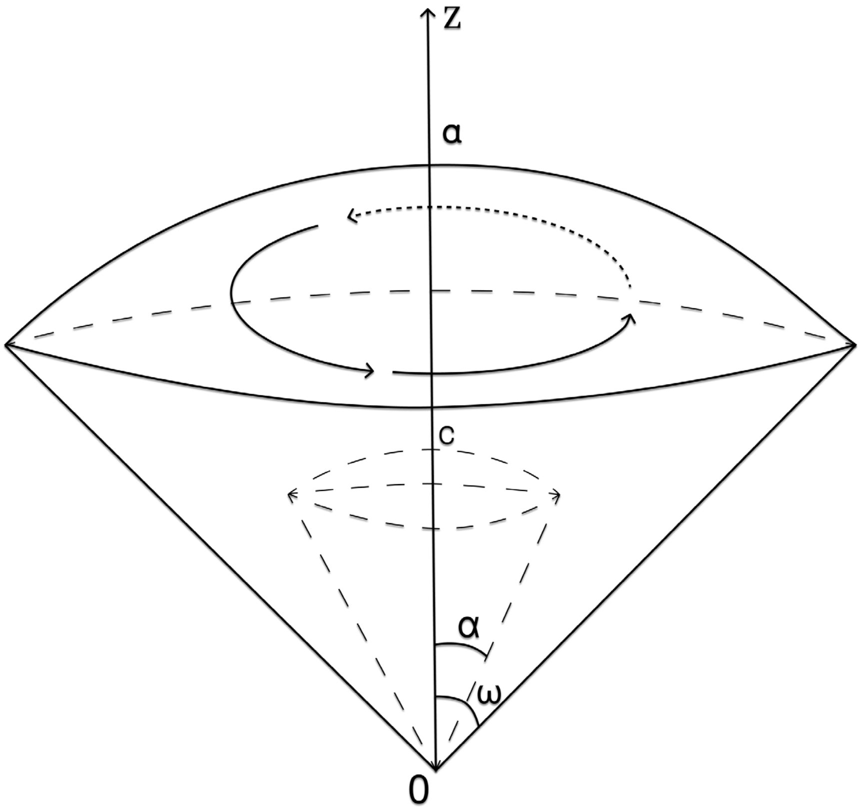

Figure 1 – A truncated cone with a spherical crack, to which an axisymmetric tangential load is applied

In the spherical coordinate system (r, θ, φ), a cone truncated along a spherical surface is considered, occupying the region $0 < r < a,\ 0 \le \theta < \omega, 0 \le \varphi < 2\pi$. An axisymmetric tangential load with intensity $p(\theta)$, which induces torsion, is applied to the spherical surface at $r = a$. The conical surface at $\theta = \omega$ is fixed. In this case, only the angular displacement $W(r, \theta)$ and two tangential stresses are non-zero:

$$\tau_{r\varphi}(r,\theta) = Gr\frac{\partial}{\partial r}\left(\frac{1}{r}W\right), \quad \tau_{\theta\varphi}(r,\theta) = Gr\left(\frac{\partial W}{\partial \theta} - ctg(\theta)W\right),$$

where G is the shear modulus of the cone material.

The displacement $W(r, \theta)$ must satisfy the following differential equation:

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial W}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(sin\ \theta\ \frac{\partial W}{\partial \theta}\right) - \frac{W}{\sin^2\theta} = 0 \qquad (1)$$

under the boundary conditions:

$$W|_{\theta=w} = 0,\ \ \tau_{r\varphi}\big|_{r=a} = \rho(\theta) \qquad (2)$$

as well as the condition $W(0, \theta) = 0$ at the cone apex.

A spherical crack exists in the region $r = c,\ 0 \leq \theta < \alpha$, along which the edges are free of stresses. Upon crossing the crack, the displacement $W(r,\theta)$ experiences an unknown jump $f(\theta)$, and the stresses $\tau_{r\varphi}$ on the crack edges are zero. This leads to the following conditions:

$$<W> \ = W(c-0,\theta) - W(c+0,\theta) = f(\theta),\ 0 < \theta < \alpha, \tag{3}$$

where

$$f(\theta) = 0,\ \alpha < \theta < \omega \text{ and } \tau_{r\varphi}\big|_{r=c\pm 0} = 0,\ 0 < \theta < \alpha, \tag{4}$$

from which it follows that

$$<\tau_{r\varphi}> \ = \tau_{r\varphi}(c-0,\theta) - \tau_{r\varphi}(c+0,\theta) = 0. \tag{5}$$

In addition, the crack closure condition must be satisfied

$$f(\alpha) = 0 \tag{6}$$

2. Reduction of the Given Problem to a One-Dimensional Discontinuous Boundary Problem

To reduce the formulated problem (1)–(5) to a one-dimensional problem, we use the Legendre integral transformations introduced by G.Ya. Popov [6]:

$$W_k(r) = \int_0^{\omega} W(r,\theta) P_{\nu_k}^1(\cos\theta)\sin\theta\, d\theta$$

with the inverse transformation formula:

$$W(r,\theta) = \sum_{k=0}^{\infty} W_k(r) \frac{P_{\nu_k}^1(\cos\theta)}{\left\|P_{\nu_k}^1(\cos\theta)\right\|^2}$$

where ν_k are the non-negative roots of the equation $P_\nu^1(\cos\omega) = 0$.

The application of this integral transformation is described in Appendix 1. In the transformed domain, the following one-dimensional discontinuous boundary problem is obtained:

$$r^2 W_k''(r) + 2r W_k'(r) - \nu_k(\nu_k+1) W_k(r) = 0,\ \ 0 < r < a,\ \ r \neq c$$

$$W_k'(a) - \frac{1}{a} W_k(a) = \frac{P_k}{G};\ W_k(0) = 0$$

$$<W_k> \ = W_k(c-0) - W_k(c+0) = f_k$$

$$<\tau_{r\varphi,k}> \ = \tau_{r\varphi,k}(c-0) - \tau_{r\varphi,k}(c+0) = 0,$$

where

$$f_k = \int_0^{\alpha} f(\theta) P_{\nu_k}^1(\cos\theta)\sin\theta\, d\theta$$

and

$$\tau_{r\varphi,k}(r) = Gr\left(\frac{1}{r}W_k(r)\right)' = G\left(W_k{}'(r) - \frac{1}{r}W_k(r)\right)$$

3. Solution of the One-Dimensional Discontinuous Boundary Problem

The solution to the one-dimensional discontinuous boundary problem will be constructed as the sum of a continuous solution $U_k(r)$ and a discontinuous solution $V_k(r)$:

$$W_k(r) = U_k(r) + V_k(r)$$

The continuous solution $U_k(r)$ is the solution of the boundary problem:

$$r^2 U_k{}''(r) + 2rU_k{}'(r) - \nu_k(\nu_k+1)U_k(r) = 0, \;\; 0 < r < a$$

$$U_k{}'(a) - \frac{1}{a}U_k(a) = \frac{P_k}{G}; \;\; U_k(0) = 0$$

and has the form:

$$U_k(r) = \frac{aP_k}{G(\nu_k - 1)}\left(\frac{r}{a}\right)^{\nu_k}, \text{where } \; 0 < r < a \qquad (7)$$

The discontinuous solution $V_k(r)$ is the solution to the discontinuous boundary problem:

$$r^2 V_k{}''(r) + 2rV_k{}'(r) - \nu_k(\nu_k+1)V_k(r) = 0, \;\; 0 < r < a, \;\; r \neq c$$

$$V_k{}'(a) - \frac{1}{a}V_k(a) = 0; \;\; V_k(0) = 0$$

$$\langle V_k \rangle = V_k(c-0) - V_k(c+0) = f_k$$

$$\langle \tau_{r\varphi,k} \rangle = \tau_{r\varphi,k}(c-0) - \tau_{r\varphi,k}(c+0) = 0,$$

To find the discontinuous solution, we will find the Green's function $G_k(r,\rho)$ of the boundary problem:

$$r^2 V_k{}''(r) + 2rV_k{}'(r) - \nu_k(\nu_k+1)V_k(r) = 0, \;\; 0 < r < a$$

$$aV_k{}'(a) - V_k(a) = 0; \; V_k(0) = 0$$

The Green's function has the form (as derived in Appendix 2):

$$G_k(r,\rho) = -\frac{\nu_k + 2}{a(\nu_k - 1)(2\nu_k + 1)}\left(\frac{r\rho}{a^2}\right)^{\nu_k} - \frac{1}{2\nu_k + 1}\begin{cases}\frac{1}{\rho}\left(\frac{r}{\rho}\right)^{\nu_k}, & 0 < r < \rho < a \\ \frac{1}{r}\left(\frac{\rho}{r}\right)^{\nu_k}, & 0 < \rho < r < a\end{cases}$$

It can be shown that the constructed Green's function $G_k(r,\rho)$ has the following properties [18]: $G_k(r,c)$ and $\left.\frac{\partial^2 G_k(r,\rho)}{\partial r \partial \rho}\right|_{\rho=c}$ are continuous when passing from point

$r = c - 0$ to point $r = c + 0$, while the derivatives $\frac{\partial G_k(r,c)}{\partial r}$ and $\left.\frac{\partial G_k(r,\rho)}{\partial \rho}\right|_{\rho=c}$ experience jumps of $-\frac{1}{c^2}$ and $\frac{1}{c^2}$, respectively, during this transition. Thus, the discontinuous solution has the form:

$$V_k(r) = c^2 \left[<V_k(c)> \left.\frac{\partial G_k(r,\rho)}{\partial \rho}\right|_{\rho=c} - <V_k(c)> G_k(r,c) \right]$$

From the conditions on the defect, we obtain:

$$\langle V_k(c) \rangle = V_k(c-0) - V_k(c+0) = f_k$$

$$<\tau_{r\varphi,k}> = \tau_{r\varphi,k}(c-0) - \tau_{r\varphi,k}(c+0) =$$

$$= G\left[V_k{}'(c-0) - \frac{1}{c}V_k(c-0)\right] - G\left[V_k{}'(c+0) - \frac{1}{c}V_k(c+0)\right] =$$

$$= G\left[\langle V_k{}'(c) \rangle - \frac{1}{c}\langle V_k(c) \rangle\right] = 0$$

From this, it follows that $\langle V_k{}'(c) \rangle = \frac{1}{c}\ \langle V_k(c) \rangle = \frac{1}{c} f_k$. Therefore, the discontinuous solution will be:

$$V_k(r) = c^2 \left[f_k \left.\frac{\partial G_k(r,\rho)}{\partial \rho}\right|_{\rho=c} - \frac{1}{c} f_k G_k(r,c) \right] = c^2 \left[\left.\frac{\partial G_k(r,\rho)}{\partial \rho}\right|_{\rho=c} - \frac{1}{c} G_k(r,c) \right] f_k$$

The solution to the discontinuous boundary problem is the sum of the continuous and discontinuous solutions:

$$W_k(r) = \frac{aP_k}{G(\nu_k - 1)}\left(\frac{r}{a}\right)^{\nu_k} + c^2 \left[\left.\frac{\partial G_k(r,\rho)}{\partial \rho}\right|_{\rho=c} - \frac{1}{c} G_k(r,c) \right] f_k$$

Next, using the inverse transformation formula, we obtain:

$$W(r,\theta) = \frac{a}{G}\sum_{k=1}^{\infty} \frac{P_k}{\nu_k - 1}\left(\frac{r}{a}\right)^{\nu_k} \frac{P_{\nu_k}^1(\cos\theta)}{\left\|P_{\nu_k}^1(\cos\theta)\right\|^2} +$$

$$+\sum_{k=1}^{\infty} c^2 \left[\left.\frac{\partial G_k(r,\rho)}{\partial \rho}\right|_{\rho=c} - \frac{1}{c} G_k(r,c) \right] f_k \frac{P_{\nu_k}^1(\cos\theta)}{\left\|P_{\nu_k}^1(\cos\theta)\right\|^2} \quad (8)$$

The obtained expression for the displacement contains the unknown value f_k, the transform of the displacement jump on the crack. Based on condition (4) on the crack, we derive an integral equation to determine this jump.

4. Obtaining the Integral Equation

Consider

$$f_k = \int_0^\alpha f(\eta) P_{\nu_k}^1(\cos\eta)\sin\eta\, d\eta$$

and integrate by parts

$$f_k = f(\eta)\sin\eta \cdot P_{\nu_k}^1(\cos\eta)\Big|_0^\alpha - \int_0^\alpha \frac{d}{d\eta}\left(f(\eta)\sin\eta\right) P_{\nu_k}^0(\cos\eta)\, d\eta$$

Due to the crack closure condition (6), the non-integral term vanishes. Let us denote the new unknown function as:

$$g(\eta) = \frac{d}{d\eta}\left(f(\eta)\sin\eta\right)$$

As a result, we have:

$$f_k = \int_0^\alpha g(\eta) P_{\nu_k}^0(\cos\eta)\, d\eta$$

We substitute this representation for f_k into expression (8) for $W(r,\theta)$ and change the order of summation and integration (the series converges uniformly for $r \neq c$):

$$W(r,\theta) = \frac{a}{G}\sum_{k=1}^{\infty} \frac{P_k}{\nu_k - 1}\left(\frac{r}{a}\right)^{\nu_k} \frac{P_{\nu_k}^1(\cos\theta)}{\left\|P_{\nu_k}^1(\cos\theta)\right\|^2} -$$

$$-\int_0^\alpha g(\eta)\left[\sum_{k=1}^{\infty} c^2\left[\left.\frac{\partial G_k(r,\rho)}{\partial\rho}\right|_{\rho=c} - \frac{1}{c}G_k(r,c)\right]\frac{P_{\nu_k}^1(\cos\theta)P_{\nu_k}^0(\cos\eta)}{\left\|P_{\nu_k}^1(\cos\theta)\right\|^2}\right]d\eta$$

To find the unknown function $g(\eta)$, we use the condition of zero stress on the edges of the crack $\tau_{r\varphi}\big|_{r=c\pm 0} = 0,\ 0 < \theta < \alpha$. For this, we find the expression for the stress:

$$\tau_{r\varphi}(r,\theta) = Gr\frac{\partial}{\partial r}\left(\frac{1}{r}W(r,\theta)\right).$$

We apply this operator to the expressions:

$$r\frac{\partial}{\partial r}\left[\frac{1}{r}\left(\frac{r}{a}\right)^{\nu_k}\right] = \frac{\nu_k - 1}{a}\left(\frac{r}{a}\right)^{\nu_k - 1}$$

and

$$c^2 r\frac{\partial}{\partial r}\left[\frac{1}{r}\left(\frac{\partial G_k(r,\rho)}{\partial \rho}\bigg|_{\rho=c}-\frac{1}{c}G_k(r,c)\right)\right]=G_k^*(r,c)$$

where

$$G_k^*(r,c)=2c^2\frac{\nu_k+2}{2\nu_k+1}\left(\frac{r}{c}\right)^{\nu_k-1}\cdot a^{-2\nu_k-1}+\frac{(\nu_k-1)(\nu_k+2)}{2\nu_k+1}\begin{cases}\frac{1}{r}\left(\frac{r}{c}\right)^{\nu_k}, r<c\\ \frac{1}{r^2}\left(\frac{c}{r}\right)^{\nu_k}, r>c\end{cases}$$

Accordingly, we obtain:

$$\tau_{r\varphi}(r,\theta)=\frac{a}{G}\sum_{k=1}^{\infty}P_k\left(\frac{r}{a}\right)^{\nu_k-1}\cdot\frac{P_{\nu_k}^1(\cos\theta)}{\|P_{\nu_k}^1(\cos\theta)\|^2}-$$
$$-G\int_0^{\alpha}g(\eta)\left[\sum_{k=1}^{\infty}G_k^*(r,c)\cdot\frac{P_{\nu_k}^1(\cos\theta)P_{\nu_k}^0(\cos\eta)}{\|P_{\nu_k}^1(\cos\theta)\|^2}\right]d\eta$$

When $r\neq c$, as can be seen from the obtained expression for $G_k^*(r,c)$, the series converges uniformly. Therefore, we can differentiate the resulting expression with respect to θ. Taking into account that:

$$\int P_{\nu_k}^1(\cos\theta)d\theta=P_{\nu_k}^0(\cos\theta)$$

we obtain:

$$\int\tau_{r\varphi}(r,\theta)d\theta=\sum_{k=1}^{\infty}P_k\left(\frac{r}{a}\right)^{\nu_k-1}\frac{P_{\nu_k}^0(\cos\theta)}{\|P_{\nu_k}^1(\cos\theta)\|^2}-$$
$$-G\int_0^{\alpha}g(\eta)\left[\sum_{k=1}^{\infty}G_k^*(r,c)\frac{P_{\nu_k}^0(\cos\theta)P_{\nu_k}^0(\cos\eta)}{\|P_{\nu_k}^1(\cos\theta)\|^2}\right]d\eta+A(r)$$

where $A(r)$ is some function.

To derive the integral equation for $g(\eta)$, we should take the limit as $r\to c\pm 0$ and set the resulting expression equal to zero. Using the formula for $G_k^*(r,c)$, we obtain:

$$\sum_{k=1}^{\infty}P_k\left(\frac{c}{a}\right)^{\nu_k-1}\cdot\frac{P_{\nu_k}^0(\cos\theta)}{\|P_{\nu_k}^1(\cos\theta)\|^2}-$$
$$-G\int_0^{\alpha}g(\eta)\left[2c^2\sum_{k=1}^{\infty}\frac{\nu_k+2}{2\nu_k+1}a^{-2\nu_k-1}\cdot\frac{P_{\nu_k}^0(\cos\theta)P_{\nu_k}^0(\cos\eta)}{\|P_{\nu_k}^1(\cos\theta)\|^2}+\right.$$

$$+\frac{1}{c}\sum_{k=1}^{\infty}\frac{(\nu_k-1)(\nu_k+2)}{2\nu_k+1}\cdot\frac{P_{\nu_k}^0(\cos\theta)P_{\nu_k}^0(\cos\eta)}{\left\|P_{\nu_k}^1(\cos\theta)\right\|^2}\Bigg]d\eta+A_0=0,\ \ 0\le\theta<\alpha,$$

where $A_0 = A(c)$ is an unknown constant.

We now examine the convergence of the obtained series using the known asymptotics of the Legendre functions [19]:

$$P_{\nu_k}^0(\cos\theta)\sim\sqrt{\frac{2}{\pi\sin\theta}}\cdot\nu_k^{-\frac{1}{2}}\cos\left[\left(\nu_k+\frac{1}{2}\right)\theta-\frac{\pi}{4}\right],\ \ k\to\infty$$

$$P_{\nu_k}^0(\cos\eta)\sim\sqrt{\frac{2}{\pi\sin\eta}}\cdot\nu_k^{-\frac{1}{2}}\cos\left[\left(\nu_k+\frac{1}{2}\right)\eta-\frac{\pi}{4}\right],\ \ k\to\infty$$

as well as the asymptotics obtained in Appendix 3 for

$$\nu_k\sim\frac{\pi}{\omega}k;\ \ \left\|P_{\nu_k}^1(\cos\theta)\right\|^2\sim\frac{\omega}{\pi}\nu_k\,,\ \ k\to\infty \tag{9}$$

The series

$$\sum_{k=1}^{\infty}P_k\left(\frac{c}{a}\right)^{\nu_k-1}\cdot\frac{P_{\nu_k}^0(\cos\theta)}{\left\|P_{\nu_k}^1(\cos\theta)\right\|^2}$$

will converge uniformly, as the condition $c<\alpha$, and considering (9)

$$\frac{P_{\nu_k}^0(\cos\theta)}{\left\|P_{\nu_k}^1(\cos\theta)\right\|^2}\sim\frac{\pi}{\omega}\sqrt{\frac{2}{\pi\sin\theta}}\cdot\nu_k^{-\frac{3}{2}}\cos\left[\left(\nu_k+\frac{1}{2}\right)\theta-\frac{\pi}{4}\right],\ \ k\to\infty$$

Let us denote its sum as:

$$M(\theta)=\sum_{k=1}^{\infty}P_k\left(\frac{c}{a}\right)^{\nu_k-1}\cdot\frac{P_{\nu_k}^0(\cos\theta)}{\left\|P_{\nu_k}^1(\cos\theta)\right\|^2}$$

Next, the series

$$\sum_{k=1}^{\infty}\frac{\nu_k+2}{2\nu_k+1}\cdot a^{-2\nu_k-1}\cdot\frac{P_{\nu_k}^0(\cos\theta)P_{\nu_k}^0(\cos\eta)}{\left\|P_{\nu_k}^1(\cos\theta)\right\|^2}$$

will also converge uniformly, as the asymptotic of its general term is as follows

$$\frac{a^{-2\nu_k-1}}{\omega\sqrt{\sin\theta\cdot\sin\eta}}\cdot\nu_k^{-2}\cos\left[\left(\nu_k+\frac{1}{2}\right)\theta-\frac{\pi}{4}\right]\cos\left[\left(\nu_k+\frac{1}{2}\right)\eta-\frac{\pi}{4}\right],\ \ k\to\infty$$

Let us denote its sum as:

$$R(\theta,\eta)=\sum_{k=1}^{\infty}\frac{\nu_k+2}{2\nu_k+1}\cdot a^{-2\nu_k-1}\cdot\frac{P_{\nu_k}^0(\cos\theta)P_{\nu_k}^0(\cos\eta)}{\left\|P_{\nu_k}^1(\cos\theta)\right\|^2}$$

As for the remaining series

$$\sum_{k=1}^{\infty} \frac{(\nu_k - 1)(\nu_k + 2)}{2\nu_k + 1} \cdot \frac{P_{\nu_k}^0(\cos\theta) P_{\nu_k}^0(\cos\eta)}{\left\|P_{\nu_k}^1(\cos\theta)\right\|^2}$$

the asymptotic of its general term

$$\frac{2}{\omega\sqrt{\sin\theta \cdot \sin\eta}} \cdot {\nu_k}^{-1} \cos\left[\left(\nu_k + \frac{1}{2}\right)\theta - \frac{\pi}{4}\right] \cos\left[\left(\nu_k + \frac{1}{2}\right)\eta - \frac{\pi}{4}\right], \; k \to \infty$$

implies conditional convergence.

In Appendix 4, the extraction and summation of its slowly converging part has been carried out, and it has been brought to the form:

$$\frac{1}{2\omega\sqrt{\sin\theta \sin\eta}}\left[-\ln\frac{1}{|\theta-\eta|} + \ln\frac{2\sin\frac{\pi}{2\omega}|\theta-\eta|}{|\theta-\eta|} + \frac{\omega-(\theta+\eta)}{2}\right] + N(\theta,\eta),$$

where

$$N(\theta,\eta) = \sum_{k=1}^{N}\left[\frac{(\nu_k - 1)(\nu_k + 2)}{2\nu_k + 1} \cdot \frac{P_{\nu_k}^0(\cos\theta) P_{\nu_k}^0(\cos\eta)}{\left\|P_{\nu_k}^1(\cos\theta)\right\|^2} - \right.$$

$$\left. - \frac{1}{2\omega\sqrt{\sin\theta \sin\eta}}\left(\cos\nu_k(\theta+\eta) + \sin\nu_k(\theta+\eta)\right)\right],$$

and

$$\lim_{|\theta - r| \to 0} \ln\frac{2\sin\frac{\pi}{2\omega}|\theta-\eta|}{|\theta-\eta|} = \ln\frac{\pi}{2\omega}$$

Thus, the integral equation for the function $g(\eta)$ takes the form:

$$\frac{G}{2\omega c\sqrt{\sin\theta}}\int_0^\alpha \frac{g(\eta)}{\sqrt{\sin\eta}}\ln\frac{1}{|\theta-\eta|}d\eta - G\int_0^\alpha g(\eta)\left[2c^2 R(\theta,\eta) + \frac{1}{c}N(\theta,\eta) + \right.$$

$$\left. + \frac{1}{2\omega\sqrt{\sin\theta\sin\eta}}\left(\ln\frac{2\sin\frac{\pi}{2\omega}|\theta-\eta|}{|\theta-\eta|} + \frac{\omega-(\theta+\eta)}{2}\right)\right]d\eta =$$

$$= -M(\theta) - A_0, \qquad 0 \le \theta < \alpha$$

Multiplying the resulting equation $2\omega c\sqrt{\sin\theta}$ and introducing a new unknown function:

$$\varphi(\eta) = \frac{g(\eta)}{\sqrt{\sin\eta}}$$

the integral equation becomes:

$$\int_0^\alpha \varphi(\eta)\ln\frac{1}{|\theta-\eta|}d\eta - \int_0^\alpha \varphi(\eta)K(\theta,\eta)d\eta =$$

$$= -2\omega c G^{-1}\sqrt{\sin\theta}\,(M(\theta)+A_0),\quad 0\le\theta<\alpha \qquad (10)$$

where $K(\theta,\eta)$ can be written as:

$$K(\theta,\eta) = 2\omega c\sqrt{\sin\theta\sin\eta}\left[2c^2R(\theta,\eta)+\frac{1}{c}N(\theta,\eta)\right] + \ln\frac{2\sin\frac{\pi}{2\omega}|\theta-\eta|}{|\theta-\eta|} +$$

$$+\frac{\omega-(\theta+\eta)}{2}$$

5. Solving the Integral Equation by the Method of Orthogonal Polynomials

In the obtained integral equation (10), we first perform a change of variables to transform the integration interval from $(0;\alpha)$ to the interval $(-1;1)$:

$$\theta = \frac{\alpha}{2}(t+1);\quad \eta = \frac{\alpha}{2}(\xi+1)$$

As a result, we arrive at the equation:

$$\int_{-1}^{1}\varphi^*(\xi)\ln\frac{1}{|t-\xi|}d\xi - \int_{-1}^{1}\varphi^*(\xi)\left[K^*(t,\xi)-\ln\frac{2}{\alpha}\right]d\xi =$$

$$= -\frac{4\omega c}{\alpha G}\sqrt{\sin\frac{\alpha}{2}(t+1)}\,(M^*(t)+A_0)$$

where

$$\varphi^*(\xi) = \varphi\left(\frac{\alpha}{2}(\xi+1)\right);$$

$$M^*(t) = M\left(\frac{\alpha}{2}(t+1)\right);$$

$$K^*(t,\xi) = K\left(\frac{\alpha}{2}(t+1),\frac{\alpha}{2}(\xi+1)\right);$$

The presence of the spectral relation [18]:

$$\frac{1}{\pi}\int_{-1}^{1}\ln\frac{1}{|x-y|}\cdot\frac{T_n(y)}{\sqrt{1-y^2}}dy = \begin{cases}\ln 2, n=0\\ \frac{1}{\pi}T_n(x),\quad n=1,2,\ldots\end{cases}$$

allows the use of the method of orthogonal polynomials [18] to solve the obtained integral equation. We will seek the solution in the form of the expansion of the desired function into a series of Chebyshev polynomials of the first kind $T_n(\xi)$

$$\varphi^*(\xi)=\frac{1}{\sqrt{1-\xi^2}}\sum_{k=0}^{\infty}\varphi_k^*T_k(\xi)$$

Let us substitute the expansion for $\varphi^*(\xi)$ into the integral equation and use the spectral relation:

$$\sum_{k=0}^{\infty}\varphi_k^*\int_{-1}^{1}\ln\frac{1}{|t-\xi|}\cdot\frac{T_k(\xi)}{\sqrt{1-\xi^2}}d\xi-\sum_{k=0}^{\infty}\varphi_k^*\int_{-1}^{1}\frac{T_k(\xi)}{\sqrt{1-\xi^2}}\left[K^*(t,\xi)-\ln\frac{2}{\alpha}\right]d\xi=$$

$$=-\frac{4\omega c}{\alpha G}\sqrt{\sin\frac{\alpha}{2}(t+1)}\,(M^*(t)+A_0)$$

$$\sum_{k=0}^{\infty}\gamma_k\varphi_k^*T_k(t)-\sum_{k=0}^{\infty}\varphi_k^*\int_{-1}^{1}\frac{T_k(\xi)}{\sqrt{1-\xi^2}}K^*(t,\xi)d\xi+\ln\frac{2}{\alpha}\sum_{k=0}^{\infty}\varphi_k^*\int_{-1}^{1}\frac{T_k(\xi)}{\sqrt{1-\xi^2}}d\xi=$$

$$=-\frac{4\omega c}{\alpha G}\sqrt{\sin\frac{\alpha}{2}(t+1)}\,(M^*(t)+A_0),$$

where

$$\gamma_k=\begin{cases}\pi\ln 2\,, k=0\\ \dfrac{\pi}{k},\quad k\geq 1\end{cases}$$

Taking into account the orthogonality conditions of the Chebyshev polynomials:

$$\int_{-1}^{1}\frac{T_k(\xi)T_j(\xi)}{\sqrt{1-\xi^2}}d\xi=\delta_{kj}|T_k(\xi)|^2,\ |T_k(\xi)|^2=\begin{cases}\pi, & k=0\\ \dfrac{\pi}{2}, & k\geq 1\end{cases},\ \delta_{kj}=\begin{cases}1, & k=j\\ 0, & k\neq j\end{cases}$$

and the fact that $T_0(\xi)=1$, we get:

$$\int_{-1}^{1}\frac{T_k(\xi)}{\sqrt{1-\xi^2}}d\xi=\pi\delta_{k0}$$

that is,

$$\ln\frac{2}{\alpha}\sum_{k=0}^{\infty}\varphi_k^*\int_{-1}^{1}\frac{T_k(\xi)}{\sqrt{1-\xi^2}}d\xi=\pi\ln\frac{2}{\alpha}\varphi_0^*$$

Let us multiply the obtained equation by $\frac{T_j(t)}{\sqrt{1-t^2}}$ and integrate with respect to t

from -1 to 1:

$$\sum_{k=0}^{\infty}\gamma_k\varphi_k^*\int_{-1}^{1}\frac{T_k(t)T_j(t)}{\sqrt{1-t^2}}dt-\sum_{k=0}^{\infty}\varphi_k^*\int_{-1}^{1}\frac{T_j(t)}{\sqrt{1-t^2}}dt\int_{-1}^{1}\frac{T_k(\xi)}{\sqrt{1-\xi^2}}K^*(t,\xi)d\xi+$$

$$+\ln\frac{2}{\alpha}\varphi_0^*\int_{-1}^{1}\frac{T_j(t)}{\sqrt{1-t^2}}dt=-\frac{4\omega c}{\alpha G}\int_{-1}^{1}\frac{T_j(t)}{\sqrt{1-t^2}}\sin\sqrt{\frac{\alpha}{2}(t+1)}\,(M^*(t)+A_0)dt$$

Let us apply the orthogonality condition:

$$\pi^2B_j\varphi_j^*-\sum_{k=0}^{\infty}\varphi_k^*B_{jk}+\pi^2\ln\frac{2}{\alpha}\varphi_0^*\delta_{j0}=C_j+A_0D_j,\ \text{ where } j=0,1,\dots$$

$$B_j=\begin{cases}\ln 2\,, j=0\\ \dfrac{1}{j}, \quad j\geq 1\end{cases}$$

$$B_{jk}=\int_{-1}^{1}\frac{T_j(t)}{\sqrt{1-t^2}}dt\int_{-1}^{1}\frac{T_k(\xi)}{\sqrt{1-\xi^2}}K^*(t,\xi)d\xi\,;$$

$$C_j=-\frac{4\omega c}{\alpha G}\int_{-1}^{1}\frac{T_j(t)}{\sqrt{1-t^2}}M^*(t)\sin\sqrt{\frac{\alpha}{2}(t+1)}\,dt;$$

$$D_j=-\frac{4\omega c}{\alpha G}\int_{-1}^{1}\frac{T_j(t)}{\sqrt{1-t^2}}\sin\sqrt{\frac{\alpha}{2}(t+1)}\,dt;$$

We normalize the resulting infinite system of linear algebraic equations. Divide by π^2 and $\sqrt{B_j}$, and introduce $\psi_j=\sqrt{B_j}\varphi_j^*;\ \varphi_j^*=\frac{1}{\sqrt{B_j}}\psi_j$. Then

$$\psi_j-\sum_{k=0}^{\infty}B_{jk}^*\psi_k+\frac{ln\frac{2}{\alpha}}{ln\,2}\psi_0\delta_{j0}=C_j^*+A_0D_j^*,\ \text{ где } j=0,1,\dots$$

$$B_{jk}^*=\frac{B_{jk}}{\sqrt{B_jB_k}};\ C_j^*=\frac{C_j}{\sqrt{B_jB_k}};\ D_j^*=\frac{D_j}{\sqrt{B_jB_k}}$$

To compute integrals of the form:

$$I(j) = \int_{-1}^{1} F(t)\frac{T_j(t)}{\sqrt{1-t^2}}dt$$

we will use the Chebyshev quadrature formula [22]:

$$\int_{-1}^{1} f(x)\frac{dx}{\sqrt{1-x^2}} \approx \frac{\pi}{n}\sum_{i=1}^{n} f(x_i); \;\; x_i = \cos\frac{(2i-1)\pi}{2n};$$

$$T_j(x_i) = \cos(j\cos^{-1}x_i) = \cos\frac{j(2i-1)\pi}{2n}$$

Thus,

$$I(j) = \frac{\pi}{n}\sum_{i=1}^{n} F(x_i)\cos\frac{j(2i-1)\pi}{2n}$$

Since the right-hand side of the system contains the unknown constant A_0, this system needs to be solved for two right-hand sides C_j^* and D_j^*. Denote these solutions by $\widetilde{\psi_j}$ and $\widetilde{\widetilde{\psi}}_j$. Then $\psi_j = \widetilde{\psi_j} + A_0\widetilde{\widetilde{\psi}}_j$. To find A_0, let us consider the function $g(\eta) = \sqrt{sin\ \eta}\cdot\varphi(\eta)$ and compute:

$$\int_0^{\alpha} \varphi(\eta)\sqrt{\sin\eta}\,d\eta \;=\; \int_0^{\alpha} g(\eta)d\eta \;=\; \int_0^{\alpha} \frac{d}{d\eta}(f(\eta)\sin\eta)d\eta = f(\alpha)\sin\alpha = 0$$

considering the conditions of crack closure in equation (6). On the other hand,

$$\int_0^{\alpha} \varphi(\eta)\sqrt{\sin\eta}\,d\eta \;=\; \int_{-1}^{1} \varphi\left(\frac{\alpha}{2}(\xi+1)\right)\sqrt{\sin\frac{\alpha}{2}(\xi+1)}\cdot\frac{\alpha}{2}\,d\xi =$$

$$= \frac{\alpha}{2}\int_{-1}^{1} \varphi^*(\xi)\sqrt{\sin\frac{\alpha}{2}(\xi+1)}\,d\xi = \frac{\alpha}{2}\int_{-1}^{1} \frac{1}{\sqrt{1-\xi^2}}\sum_{k=0}^{\infty}\varphi_k^* T_k(\xi)\sqrt{\sin\frac{\alpha}{2}(\xi+1)}\,d\xi =$$

$$= \frac{\alpha}{2}\sum_{k=0}^{\infty}\varphi_k^*\int_{-1}^{1}\frac{T_k(\xi)}{\sqrt{1-\xi^2}}\sqrt{\sin\frac{\alpha}{2}(\xi+1)}\,d\xi =$$

$$= \frac{\alpha}{2}\sum_{k=0}^{\infty}\frac{\psi_k}{\sqrt{B_k}}\int_{-1}^{1}\frac{T_k(\xi)}{\sqrt{1-\xi^2}}\sqrt{\sin\frac{\alpha}{2}(\xi+1)}\,d\xi =$$

$$= \frac{\alpha}{2}\sum_{k=0}^{\infty}\frac{1}{\sqrt{B_k}}\left(\widetilde{\psi_j}+A_0\widetilde{\widetilde{\psi_j}}\right)\int_{-1}^{1}\frac{T_k(\xi)}{\sqrt{1-\xi^2}}\sqrt{\sin\frac{\alpha}{2}(\xi+1)}\,d\xi = 0,$$

that is,

$$\sum_{k=0}^{\infty}\frac{\widetilde{\psi_k}}{\sqrt{B_k}}\int_{-1}^{1}\frac{T_k(\xi)}{\sqrt{1-\xi^2}}\sqrt{\sin\frac{\alpha}{2}(\xi+1)}\,d\xi +$$

$$+A_0\sum_{k=0}^{\infty}\frac{\widetilde{\widetilde{\psi_j}}}{\sqrt{B_k}}\int_{-1}^{1}\frac{T_k(\xi)}{\sqrt{1-\xi^2}}\sqrt{\sin\frac{\alpha}{2}(\xi+1)}\,d\xi = 0$$

From this, we find A_0:

$$A_0 = -\left[\sum_{k=0}^{\infty}\frac{\widetilde{\psi_k}}{\sqrt{B_k}}\int_{-1}^{1}\frac{T_k(\xi)}{\sqrt{1-\xi^2}}\sqrt{\sin\frac{\alpha}{2}(\xi+1)}\,d\xi\right]$$

$$\cdot\left[\sum_{k=0}^{\infty}\frac{\widetilde{\widetilde{\psi_j}}}{\sqrt{B_k}}\int_{-1}^{1}\frac{T_k(\xi)}{\sqrt{1-\xi^2}}\sqrt{\sin\frac{\alpha}{2}(\xi+1)}\,d\xi\right]^{-1}$$

6. Determination of the Stress Intensity Factor

In the analysis of elasticity problems for bodies containing defects in the form of cracks or inclusions, a key focus is on the stress intensity factor (SIF). Its value is involved in fracture criteria, particularly when studying the growth of cracks. As the crack tip is approached from the outside, the stress increases indefinitely and the SIF serves as a measure of stress concentration near the crack tip. In the case of torsion problems for bodies, the SIF is defined as:

$$K_{III} = \lim_{\theta\to\alpha+0}\sqrt{2\pi(\theta-\alpha)}\cdot\tau_{r\varphi}(r,\theta)\big|_{r=c}$$

Let us write the previously derived expression for the shear stress in the form:

$$\tau_{r\varphi}(r,\theta)\big|_{r=c} = -\frac{G}{2\omega c}\frac{d}{d\theta}\left[\frac{1}{\sqrt{\sin\theta}}\int_{0}^{\alpha}\varphi(r)\ln\frac{1}{|\theta-r|}\,dr\right]+\cdots$$

where the ellipsis represents terms that have a finite limit as $\theta\to\alpha+0$, since at this limit, these terms approach zero due to the factor $\sqrt{2\pi(\theta-\alpha)}$.

Taking into account the previously applied change of variables, we obtain:

$$\tau_{r\varphi}(r,\theta)\big|_{r=c} = -\frac{G}{2\omega c}\frac{d}{dt}\left[\frac{1}{\sqrt{\sin\frac{\alpha}{2}(t+1)}}\int_{-1}^{1}\varphi^*(t)ln\frac{1}{|t-\xi|}d\xi\right] + \cdots =$$

$$= -\frac{G}{2\omega c}\left[-\frac{\alpha}{4}\cos\frac{\alpha}{2}(t+1)\left(\sin\frac{\alpha}{2}(t+1)\right)^{-\frac{3}{2}}\cdot\int_{-1}^{1}\varphi^*(\xi)\ln\frac{1}{|t-\xi|}d\xi\right.$$

$$\left.+\left(\sin\frac{\alpha}{2}(t+1)\right)^{-\frac{1}{2}}\cdot\frac{d}{dt}\int_{-1}^{1}\varphi^*(\xi)\ln\frac{1}{|t-\xi|}d\xi\right] + \cdots$$

The expression for the stress intensity factor (SIF) will take the form:

$$K_{III} = \lim_{t\to 1+0}\sqrt{\alpha\pi(t+1)}\cdot\frac{G}{2\omega c}\left[\frac{\alpha}{4}\cos\frac{\alpha}{2}(t+1)\left(\sin\frac{\alpha}{2}(t+1)\right)^{-\frac{3}{2}}\cdot\right.$$

$$\left.\cdot\int_{-1}^{1}\varphi^*(\xi)\ln\frac{1}{|t-\xi|}d\xi - \left(\sin\frac{\alpha}{2}(t+1)\right)^{-\frac{1}{2}}\cdot\frac{d}{dt}\int_{-1}^{1}\varphi^*(\xi)\ln\frac{1}{|t-\xi|}d\xi\right]$$

Substitute into this expression:

$$\varphi^*(\xi) = \frac{1}{\sqrt{1-\xi^2}}\sum_{k=0}^{\infty}\varphi_k^* T_k(\xi)$$

and we obtain:

$$K_{III} = \frac{G\sqrt{\alpha\pi}}{2\omega c}\lim_{t\to t+0}\sqrt{t-1}\left[\frac{\alpha}{4}\cos\frac{\alpha}{2}(t+1)\left(\sin\frac{\alpha}{2}(t+1)\right)^{-\frac{3}{2}}\cdot\right.$$

$$\cdot\sum_{k=0}^{\infty}\varphi_k^*\int_{-1}^{1}\ln\frac{1}{|t-\xi|}\cdot\frac{T_k(\xi)}{\sqrt{1-\xi^2}}d\xi -$$

$$\left.-\sqrt{\sin\frac{\alpha}{2}(\xi+1)}\sum_{k=0}^{\infty}\varphi_k^*\frac{d}{dt}\int_{-1}^{1}\ln\frac{1}{|t-\xi|}\cdot\frac{T_k(\xi)}{\sqrt{1-\xi^2}}d\xi\right]$$

We will use the following result [21]:

$$\int_{-1}^{1}\ln\frac{1}{|t-\xi|}\cdot\frac{T_k(\xi)}{\sqrt{1-\xi^2}}d\xi \to \frac{\pi}{k},\quad \text{as } t\to 1+0$$

$$\frac{d}{dt}\int_{-1}^{1}\ln\frac{1}{|t-\xi|}\cdot\frac{T_k(\xi)}{\sqrt{1-\xi^2}}d\xi \to -\frac{\pi\sqrt{2}}{2\sqrt{t-1}}+2\pi k,\ \text{ as } t\to 1+0$$

Then we can take the limit and use the formulas provided earlier:

$$K_{III}=\frac{G\sqrt{\alpha\pi}}{2\omega c}\cdot\left[\frac{\alpha}{4}\cos\alpha(\sin\alpha)^{-\frac{3}{2}}\sum_{k=0}^{\infty}\varphi_k^*\lim_{t\to1+0}\sqrt{t-1}\int_{-1}^{1}\ln\frac{1}{|t-\xi|}\cdot\frac{T_k(\xi)}{\sqrt{1-\xi^2}}d\xi-\right.$$

$$\left.-\sqrt{\sin\alpha}\sum_{k=0}^{\infty}\varphi_k^*\lim_{t\to1+0}\sqrt{t-1}\frac{d}{dt}\int_{-1}^{1}\ln\frac{1}{|t-\xi|}\cdot\frac{T_k(\xi)}{\sqrt{1-\xi^2}}d\xi\right]=$$

$$=-\frac{G\sqrt{\alpha\pi}}{\sqrt{\sin\alpha}}\sum_{k=0}^{\infty}\varphi_k^*\lim_{t\to1+0}\sqrt{t-1}\cdot\left[-\frac{\pi\sqrt{2}}{2\sqrt{t-1}}\right]=\frac{\pi G}{4\omega c}\cdot\sqrt{\frac{2\alpha\pi}{\sin\alpha}}\sum_{k=0}^{\infty}\varphi_k^*=$$

$$=\frac{\pi G}{4\omega c}\cdot\sqrt{\frac{2\alpha\pi}{\sin\alpha}}\sum_{k=0}^{\infty}\frac{\psi_k}{\sqrt{B_k}}$$

7. Numerical Results

This section presents the numerical results of the boundary value problem for a truncated cone with a crack.

Table 1 - Roots of the equation $P_\nu^1(\cos\omega)=0$

ν_k / ω	$\pi/3$	$\pi/4$	$\pi/6$
ν_0	0	0	0
ν_1	3.19569115	4.405329182	6.835398076
ν_2	6.21952915	8.447112619	12.90828411
ν_3	9.22884936	12.46332876	18.93644579
ν_4	12.23842913	16.43193967	24.99134578
ν_5	15.24800891	20.39145199	30.04624577
ν_6	18.25858867	24.35096421	36.10114576
ν_7	21.26916844	28.31047633	42.15604575
ν_8	24.27974821	32.26998836	48.21094574
ν_9	27.29032798	36.22950029	54.26584573

The analysis uses the eigenvalues and squared norms of Legendre functions, computed for several values of the angles θ. The obtained values are shown in tables, which display the first roots of the equation $P_{\nu}^{1}(\cos\omega) = 0$ (Table 1), and the corresponding squared norm values $\|P_{\nu_k}^{1}(\cos\theta)\|^2$ (Table 2).

Table 2 - Values of the norms $\|P_{\nu_k}^{1}(\cos\theta)\|^2$

ν_k / θ	$\pi/3$	$\pi/4$	$\pi/6$
ν_0	0	0	0
ν_1	9.58707345	17.621316728	41.012388456
ν_2	18.65858745	33.788450476	77.44970466
ν_3	27.68654808	49.85331504	113.61867474
ν_4	36.71528739	65.72775868	149.94807468
ν_5	45.74402673	81.56580796	180.27747462
ν_6	54.77576601	97.40385684	216.60687456
ν_7	63.80750532	113.24190532	252.9362745
ν_8	72.83924463	129.07995344	289.26567444
ν_9	81.87098394	144.91800116	325.59507438

As can be easily seen, the obtained values agree well with the asymptotic results presented in Appendix 3.

To analyze the stress state near the crack, the stress intensity factor (SIF) K_{III} was calculated for various geometric parameters of the problem. The applied load was chosen in the form

$$P(\theta) = P_*(\omega - \theta)^2$$

Table 3 – Influence of crack height c on K_{III} for fixed angles $\omega = \pi/3$, $\alpha = \pi/6$ with a cone height of $a = 1$

c	K_{III}
0.2	0.15566
0.4	0.76263
0.6	1.76408
0.8	3.14367

The system of infinite equations was solved numerically using a reduction method, which allowed the calculation of the SIF values for various geometric parameters.

At fixed angles $\omega = {}^{\pi}/_{3}$ и $\alpha = {}^{\pi}/_{6}$, increasing the crack height c leads to a significant increase in the value of K_{III}.

Table 4 – Influence of angles ω and α on K_{III} with a fixed crack height $c = 0.4$ and cone height $a = 1$

ω	α	K_{III}
${}^{\pi}/_{3}$	${}^{\pi}/_{6}$	0.76263
${}^{\pi}/_{4}$	${}^{\pi}/_{8}$	0.45961
${}^{\pi}/_{6}$	${}^{\pi}/_{12}$	0.19519

At a fixed crack height $c = 0.4$, increasing the angle ω and correspondingly decreasing the angle α leads to a decrease in the value of K_{III}.

8. Conclusions

In this work, the axisymmetric problem of torsion for a truncated cone with a crack was solved using the method of Legendre integral transforms. The problem was reduced to a one-dimensional boundary problem with a discontinuity, for which analytical solutions were obtained in the form of a sum of continuous and discontinuous solutions. During the solution process, eigenvalues and the roots of the equation $P_\nu^1(\cos\omega) = 0$ were found, as well as the corresponding norms $\left\|P_{\nu_k}^1(\cos\theta)\right\|^2$.

Based on the obtained expressions for displacements and stresses, an integral equation was derived to find the unknown displacement discontinuity at the crack, which was then solved using the method of orthogonal polynomials with the application of Chebyshev polynomials. As a result, precise formulas were obtained for calculating the stress intensity factors (SIF) K_{III} near the crack, which allowed for a detailed numerical analysis of the influence of the geometric parameters of the cone and the crack on the stress state.

Appendix 1 We multiply both sides of equation (1) by $P_\nu^1(\cos\theta)\sin\theta$ and integrate with respect to the variable θ from 0 to ω:

$$\int_0^\omega \frac{\partial}{\partial r}\left(r^2\frac{\partial W}{\partial r}\right)P_\nu^1(\cos\theta)\sin\theta\,d\theta +$$

$$+\int_0^\omega \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial W}{\partial\theta}\right)-\frac{1}{\sin^2\theta}W\right]P_\nu^1(\cos\theta)\sin\theta\,d\theta = 0$$

In the first integral, we change the order of integration and differentiation:

$$I_1 = \frac{\partial}{\partial r} \times \left[r^2 \frac{\partial}{\partial r} \int_0^{\omega} W P_\nu^1(\cos\theta) \sin\theta \, d\theta \right] = \frac{\partial}{\partial r}\left(r^2 \frac{\partial W}{\partial r} \right)$$

In the second integral, the first term is integrated by parts twice:

$$\int_0^{\omega} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial W}{\partial \theta} \right) P_\nu^1(\cos\theta) d\theta =$$

$$= \left. \frac{\partial W}{\partial \theta} \right|_{\theta=\omega} \sin\omega \, P_\nu^1(\cos\omega) - W|_{\theta=\omega} \sin\omega \left. \frac{\partial P_\nu^1(\cos\omega)}{\partial \theta} \right|_{\theta=\omega}$$

$$+ \int_0^{\omega} W \frac{\partial}{\partial \theta}\left(\sin\omega \frac{\partial P_\nu^1(\cos\omega)}{\partial \theta} \right) d\theta$$

Due to the first boundary condition (2), the second term vanishes, and for the vanishing of the first boundary term, we require that ν_k be the non-positive roots of the equation $P_\nu^1(\cos\omega) = 0$. The second integral takes the form:

$$\int_0^{\omega} W \left[\frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial P_{\nu_k}^1(\cos\theta)}{\partial \theta} \right) - \frac{1}{\sin^2\theta} P_{\nu_k}^1(\cos\theta) \right] \sin\theta \, d\theta$$

Using the Legendre equation:

$$\frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial P_{\nu_k}^1(\cos\theta)}{\partial \theta} \right) - \frac{1}{\sin^2\theta} P_{\nu_k}^1(\cos\theta) = \nu(\nu+1) P_{\nu_k}^1(\cos\theta)$$

this integral is written as:

$$I_2 = -\nu_k(\nu_k+1) \int_0^{\omega} W P_{\nu_k}^1(\cos\theta) \sin\theta \, d\theta = -\nu_k(\nu_k+1) W_k$$

From here, the equation in the transform will be as follows:

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial W_k}{\partial r} \right) - \nu_k(\nu_k+1) W_k = 0, \qquad 0 < r < a, \; r \neq c$$

Applying this integral transformation to the second boundary condition (2) and the condition at $r = 0$, we obtain:

$$W_k{}'(a) - \frac{1}{a} W_k(a) = \frac{P_k}{G}, \; W_k(0) = 0$$

Appendix 2 To find the discontinuous solution, we construct the Green's function in the form:

$$G_k(r,\rho) = \begin{cases} a_0 r^{\nu_k} + a_1 r^{-\nu_k - 1}, & 0 < r < \rho < a \\ b_0 r^{\nu_k} + b_1 r^{-\nu_k - 1}, & 0 < \rho < r < a \end{cases}$$

Using the conditions for the continuity of the Green's function and the jump in the first derivative, we get:

$$\begin{cases} a_0\rho^{\nu_k} + b_1\rho^{-\nu_k-1} - a_0\rho^{\nu_k} - a_1\rho^{-\nu_k-1} = 0 \\ b_0\nu_k\rho^{-\nu_k-1} - b_1(\nu_k+1)\rho^{-\nu_k-2} - a_0\nu_k\rho^{\nu_k-1} + a_1(\nu_k+1)\rho^{-\nu_k-2} = \dfrac{1}{\rho^2} \end{cases}$$

From the boundary condition $\nu_k(0) = 0$, it follows that $a_1 = 0$. From the boundary condition $a{\nu_k}'(a) - \nu_k(a) = 0$, we get:

$$a[b_0\nu_k a^{\nu_k-1} - b_1(\nu_k+1)a^{-\nu_k-2}] - [b_0 a^{\nu_k} + b_1 a^{-\nu_k-1}] = 0$$

Solving this system, we find:

$$a_0 = -\frac{\nu_k+2}{(\nu_k-1)(2\nu_k+1)}\left(\frac{\rho}{a^2}\right)^{\nu_k}\cdot\frac{1}{a} - \frac{1}{2\nu_k+1}\rho^{-\nu_k-1}$$

$$b_0 = -\frac{\nu_k+2}{(\nu_k-1)(2\nu_k+1)}\left(\frac{\rho}{a^2}\right)^{2}\cdot\frac{1}{a}$$

$$b_1 = -\frac{1}{2\nu_k+1}\rho^{\nu_k}$$

Then, the Green's function $G_k(r,\rho)$ is

$$G_k(r,\rho) = -\frac{\nu_k+2}{a(\nu_k-1)(2\nu_k+1)}\left(\frac{r\rho}{a^2}\right)^{\nu_k} - \frac{1}{2\nu_k+1}\begin{cases}\left(\dfrac{r}{\rho}\right)^{\nu_k}\dfrac{1}{\rho}, 0 < r < \rho < a \\ \left(\dfrac{\rho}{r}\right)^{\nu_k}\dfrac{1}{r}, 0 < \rho < r < a\end{cases}$$

Appendix 3 Let us consider the asymptotic expression for the Legendre functions [19]:

$$P_{\nu_k}^1(\cos\theta) \sim \sqrt{\frac{2}{\pi\sin\theta}}\cdot{\nu_k}^{-\frac{1}{2}}\cos\left[\left(\left(\nu_k+\frac{1}{2}\right)\theta - \frac{\pi}{4}\right],\quad k\to\infty$$

To obtain the asymptotic expression for ν_k, we need to equate the leading term of the cosine argument to zero. This condition gives us:

$$\left(\nu_k+\frac{1}{2}\right)\theta - \frac{\pi}{4} = \pi k,\quad k\in\mathbb{Z}$$

Considering large values of k, we can simplify the expression:

$$\nu_k \sim \frac{\pi}{\omega}k,\quad k\to\infty$$

Substituting these same formulas into the expressions for the norms squared, we get:

$$\|P_\nu^1(\cos\theta)\|^2 = \int_0^\omega [P_\nu^1(\cos\theta)]^2\sin\theta\,d\theta$$

After integrating, we find that:

$$\left\|P_{\nu_k}^1(\cos\theta)\right\|^2 \sim \frac{\omega}{\pi}\nu_k\,,\quad k\to\infty$$

Appendix 4 Consider the series

$$\sum_{k=1}^{\infty}\frac{(\nu_k-1)(\nu_k+2)}{2\nu_k+1}\cdot\frac{P_{\nu_k}^0(\cos\theta)P_{\nu_k}^0(\cos\eta)}{\left\|P_{\nu_k}^1(\cos\theta)\right\|^2}$$

The asymptotics of its general term is as follows

$$\frac{2}{\omega\sqrt{\sin\theta\cdot\sin\eta}}\cdot{\nu_k}^{-1}\cos\left[\left((\nu_k+\frac{1}{2}\right)\theta-\frac{\pi}{4}\right]\cos\left[\left((\nu_k+\frac{1}{2}\right)\eta-\frac{\pi}{4}\right],\quad k\to\infty$$

That is, the series converges conditionally. We isolate its slowly decaying part. For this, we write the obtained asymptotic as:

$$\frac{2}{2\omega\sqrt{\sin\theta\sin\eta}}\cdot{\nu_k}^{-1}\left[\cos\nu_k(\theta-\eta)+\cos\left[\nu_k(\theta+\eta)-\frac{\pi}{2}\right]\right]=$$

$$=\frac{{\nu_k}^{-1}}{2\omega\sqrt{\sin\theta\sin\eta}}\left[\cos\nu_k(\theta-\eta)+\sin\nu_k(\theta+\eta)\right]$$

We transform the series under consideration as follows:

$$\sum_{k=1}^{\infty}\frac{(\nu_k-1)(\nu_k+2)}{2\nu_k+1}\cdot\frac{P_{\nu_k}^0(\cos\theta)P_{\nu_k}^0(\eta)}{\left\|P_{\nu_k}^1(\cos\theta)\right\|^2}=$$

$$=\sum_{k=1}^{N}\frac{(\nu_k-1)(\nu_k+2)}{2\nu_k+1}\cdot\frac{P_{\nu_k}^0(\cos\theta)P_{\nu_k}^0(\cos\eta)}{\left\|P_{\nu_k}^1(\cos\theta)\right\|^2}+$$

$$+\sum_{k=1}^{\infty}\frac{(\nu_k-1)(\nu_k+2)}{2\nu_k+1}\cdot\frac{P_{\nu_k}^0(\cos\theta)P_{\nu_k}^0(\cos\eta)}{\left\|P_{\nu_k}^1(\cos\theta)\right\|^2}=$$

Assuming that N is large enough so that the second sum can be replaced by its asymptotic expression, we get:

$$=\sum_{k=1}^{N}\frac{(\nu_k-1)(\nu_k+2)}{2\nu_k+1}\cdot\frac{P_{\nu_k}^0(\cos\theta)P_{\nu_k}^0(\cos\eta)}{\left\|P_{\nu_k}^1(\cos\theta)\right\|^2}+$$

$$+\frac{1}{2\omega\sqrt{\sin\theta\sin\eta}}\sum_{k=N+1}^{\infty}\frac{\cos\nu_k(\theta-\eta)+\sin\nu_k(\theta+\eta)}{\nu_k}=$$

$$=\sum_{k=1}^{N}\frac{(\nu_k-1)(\nu_k+2)}{2\nu_k+1}\cdot\frac{P_{\nu_k}^0(\cos\theta)P_{\nu_k}^0(\cos\eta)}{\left\|P_{\nu_k}^1(\cos\theta)\right\|^2}-$$

$$-\sum_{k=1}^{N}\frac{1}{2\omega\sqrt{\sin\theta\sin\eta}}\cdot\frac{\cos\nu_k(\theta-\eta)+\sin\nu_k(\theta+\eta)}{\nu_k}+$$

$$+\frac{1}{2\omega\sqrt{\sin\theta\sin\eta}}\sum_{k=1}^{N}\frac{(\nu_k-1)(\nu_k+2)}{2\nu_k+1}\cdot\frac{P_{\nu_k}^{0}(\cos\theta)P_{\nu_k}^{0}(\cos\eta)}{\left\|P_{\nu_k}^{1}(\cos\theta)\right\|^2}+$$

$$+\frac{1}{2\omega\sqrt{\sin\theta\sin\eta}}\sum_{k=N+1}^{\infty}\frac{(\nu_k-1)(\nu_k+2)}{2\nu_k+1}\cdot\frac{P_{\nu_k}^{0}(\cos\theta)P_{\nu_k}^{0}(\cos\eta)}{\left\|P_{\nu_k}^{1}(\cos\theta)\right\|^2}=$$

$$=\sum_{k=1}^{N}\left[\frac{(\nu_k-1)(\nu_k+2)}{2\nu_k+1}\cdot\frac{P_{\nu_k}^{0}(\cos\theta)P_{\nu_k}^{0}(\cos\eta)}{\left\|P_{\nu_k}^{1}(\cos\theta)\right\|^2}-\right.$$

$$\left.-\frac{1}{2\omega\sqrt{\sin\theta\sin\eta}}\frac{\cos\nu_k(\theta-\eta)+\sin\nu_k(\theta+\eta)}{\nu_k}\right]+$$

$$+\frac{1}{2\omega\sqrt{\sin\theta\sin\eta}}\sum_{k=1}^{\infty}\frac{\cos\nu_k(\theta-\eta)+\sin\nu_k(\theta+\eta)}{\nu_k}$$

The first term is a finite sum, while the second term is a series that can be summed using formula 5.4.2.9 from the reference book [20]:

$$\sum_{k=1}^{\infty}\frac{1}{K}\begin{Bmatrix}\sin(kx+a)\\ \cos(kx+a)\end{Bmatrix}=\pm\frac{\pi-x}{2}\begin{Bmatrix}\cos a\\ \sin a\end{Bmatrix}-\ln\left(2\sin\frac{x}{2}\right)\begin{Bmatrix}\sin a\\ \cos a\end{Bmatrix},\ 0<x<2\pi, a=0$$

Thus,

$$\sum_{k=1}^{\infty}\frac{\cos\nu_k(\theta+\eta)+\sin\nu_k(\theta+\eta)}{\nu_k}=\frac{\omega}{\pi}\sum_{k=1}^{\infty}\frac{\cos\frac{\pi}{\omega}(\theta-\eta)k+\sin\frac{\pi}{\omega}(\theta-\eta)k}{k}=$$

$$=\frac{\omega}{\pi}ln\left(2\sin\frac{\pi}{2\omega}|\theta-\eta|\right)+\frac{\pi-\frac{\pi}{\omega}(\theta+\eta)}{2}\frac{\omega}{\pi}$$

We extract the singularity from the first term:

$$\ln\left(2\sin\frac{\pi}{2\omega}|\theta-\eta|\right)=-\ln\frac{1}{|\theta-\eta|}+\ln\frac{1}{|\theta-\eta|}+\ln\left(2\sin\frac{\pi}{2\omega}|\theta-\eta|\right)=$$

$$=-\ln\frac{1}{|\theta-\eta|}+\ln\frac{2\sin\frac{\pi}{2\omega}|\theta-\eta|}{|\theta-\eta|}$$

Note that the second term does not have a singularity, as:

$$\lim_{|\theta-r|\to 0}\ln\frac{2\sin\frac{\pi}{2\omega}|\theta-\eta|}{|\theta-\eta|}=\ln\frac{\pi}{2\omega}$$

Appendix 5 To compute the norm squared, we use the following equation:

$$\|P_\nu^1(\cos\theta)\|^2 = \int_0^\omega [P_\nu^1(\cos\theta)]^2 \sin\theta\, d\theta$$

Consider the Legendre equation with respect to $y(\theta) = P_\nu^1(\cos\theta)$:

$$L[y(\theta)] = -\frac{d}{d\theta}\left(\sin\theta \frac{dy(\theta)}{d\theta}\right) + \frac{1}{(\sin\theta)^2} y(\theta) = \nu(\nu+1)\sin\theta\, y(\theta),$$

and the two solutions $P_\nu^1(\cos\theta)$ и $P_\gamma^1(\cos\theta)$ when $\nu \neq \gamma$:

$$L[P_\nu^1(\cos\theta)] = \nu(\nu+1)\sin\theta P_\nu^1(\cos\theta)$$

$$L[P_\gamma^1(\cos\theta)] = \gamma(\gamma+1)\sin\theta P_\gamma^1(\cos\theta)$$

We multiply the first of these by $P_\gamma^1(\cos\theta)$, and the second by $P_\nu^1(\cos\theta)$, and subtract the results:

$$-\frac{\partial}{\partial\theta}\sin\theta \frac{\partial P_\nu^1(\cos\theta)}{\partial\theta} P_\gamma^1(\cos\theta) + \frac{\partial}{\partial\theta}\sin\theta \frac{\partial P_\gamma^1(\cos\theta)}{\partial\theta} P_\nu^1(\cos\theta) =$$

$$= [\nu(\nu+1) - \gamma(\gamma+1)]\sin\theta\, P_\nu^1(\cos\theta) P_\gamma^1(\cos\theta)$$

Integrating this identity over the interval $(0;\omega)$:

$$[\nu(\nu+1) - \gamma(\gamma+1)] \int_0^\omega P_\nu^1(\cos\theta) P_\gamma^1(\cos\theta)\sin\theta\, d\theta =$$

$$= -\int_0^\omega \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial P_\nu^1(\cos\theta)}{\partial\theta}\right) P_\gamma^1(\cos\theta) d\theta$$

$$+ \int_0^\omega \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial P_\gamma^1(\cos\theta)}{\partial\theta}\right) P_\nu^1(\cos\theta) d\theta$$

We integrate the first part by parts:

$$-\sin\theta \cdot \frac{\partial P_\nu^1(\cos\theta)}{\partial\theta} \cdot P_\gamma^1(\cos\theta)\bigg|_0^\omega + \int_0^\omega \sin\theta \cdot \frac{\partial P_\nu^1(\cos\theta)}{\partial\theta} \cdot \frac{\partial P_\gamma^1(\cos\theta)}{\partial\theta} d\theta +$$

$$+\sin\theta \cdot \frac{\partial P_\gamma^1(\cos\theta)}{\partial\theta} \cdot P_\nu^1(\cos\theta)\bigg|_0^\omega - \int_0^\omega \sin\theta \cdot \frac{\partial P_\gamma^1(\cos\theta)}{\partial\theta} \cdot \frac{\partial P_\nu^1(\cos\theta)}{\partial\theta} d\theta =$$

$$= \sin\omega \cdot \left[P_\nu^1(\cos\theta) \frac{\partial P_\gamma^1(\cos\theta)}{\partial\theta} - P_\gamma^1(\cos\theta) \frac{\partial P_\nu^1(\cos\theta)}{\partial\theta}\right]_{\theta=\omega}$$

Thus, we obtain the equality:

$$\int_0^\omega P_\nu^1(\cos\theta) P_\gamma^1(\cos\theta)\sin\theta\, d\theta =$$

$$= \frac{\sin\omega}{\nu(\nu+1)-\gamma(\gamma+1)}\left[P_\nu^1(\cos\theta)\frac{\partial P_\gamma^1(\cos\theta)}{\partial\theta} - P_\gamma^1(\cos\theta)\frac{\partial P_\nu^1(\cos\theta)}{\partial\theta}\right]_{\theta=\omega}, \gamma \neq \nu$$

For calculating the norm squared $(\gamma = \nu)$ the obtained equality cannot be directly applied, since there will be an indeterminate form on the right-hand side. Therefore, let us take $\gamma = \nu + \epsilon$ in this formula and consider the integral

$$\int_0^\omega P_\nu^1(\cos\theta)P_{\nu+\epsilon}^1(\cos\theta)\sin\theta\, d\theta =$$

$$= \frac{\sin\omega}{\nu(\nu+1)-(\nu+\epsilon)(\nu+\epsilon+1)}\left[P_\nu^1(\cos\theta)\frac{\partial P_{\nu+\epsilon}^1(\cos\theta)}{\partial\theta}\right.$$

$$\left. - P_{\nu+\epsilon}^1(\cos\theta)\frac{\partial P_\nu^1(\cos\theta)}{\partial\theta}\right]_{\theta=\omega}$$

Taking into account $\nu(\nu+1)-(\nu+\epsilon)(\nu+\epsilon+1) = -\epsilon(2\nu+1+\epsilon)$ and taking the limit as $\epsilon \to 0$, we obtain:

$$\int_0^\omega [P_\nu^1(\cos\theta)]^2 \sin\theta\, d\theta =$$

$$= \frac{\sin\omega}{2\nu+1}\left[P_\nu^1(\cos\theta)\cdot\frac{\partial}{\partial\nu}\frac{\partial P_\nu^1(\cos\theta)}{\partial\theta}\right]_{\theta=\omega} - \left[\frac{\partial P_\nu^1(\cos\theta)}{\partial\nu}\cdot\frac{\partial P_\nu^1(\cos\theta)}{\partial\theta}\right]_{\theta=\omega}$$

For $\nu = \nu_k$, considering that $P_{\nu_k}^1(\cos\omega) = 0$, we get:

$$\left\|P_{\nu_k}^1(\cos\theta)\right\|^2 = \frac{\sin\omega}{2\nu_k+1}\left[\frac{\partial P_{\nu_k}^1(\cos\theta)}{\partial\theta}\right]_{\theta=\omega}\cdot\left[\frac{\partial P_\nu^1(\cos\omega)}{\partial\nu}\right]_{\nu=\nu_k}$$

The derivative with respect to θ can be calculated using the formula:

$$\left.\frac{\partial P_{\nu_k}^1(\cos\theta)}{\partial\theta}\right|_{\theta=\omega} = \nu_k\cot\omega\cdot P_{\nu_k}^1(\cos\omega) - \frac{\nu_k+1}{\sin\omega}\cdot P_{\nu_k-1}^1(\cos\omega)$$

For the derivative with respect to ν, one should use the representation of the Legendre function via the hypergeometric function:

$$P_\nu^1(\cos\theta) = \cot\frac{\theta}{2}\sum_{j=0}^{\infty}\frac{(-\nu)_j(\nu+1)_j}{\Gamma(j)\cdot j!}\left(\sin\frac{\theta}{2}\right)^{2j}$$

where $(a)_j = a(a+1)\dots(a+j+1) = \frac{\Gamma(a+j)}{\Gamma(j)}$, $(a)_0 = 1$ is the Pochhammer symbol.

Let us consider a function of the form:

$$F(\nu) = (-\nu)_j \cdot (\nu + 1)_j = \frac{\Gamma(j-\nu)\cdot\Gamma(j+\nu+2)}{\Gamma(-\nu)\cdot\Gamma(\nu+1)}$$

We take the logarithm of it and find the derivative with respect to ν, considering that

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}, \quad z \neq 0, -1, -2, \dots$$

From here, we obtain:

$$F'(\nu) = F(\nu)[\psi(j+\nu+1) - \psi(\nu+1-j)]$$

As a result, we have the exact formula for calculating the derivative with respect to the index:

$$\frac{\partial P_\nu^1(\cos\theta)}{\partial\nu} = \cot\frac{\theta}{2}\sum_{j=0}^{\infty}\frac{-\nu_j(\nu+1)_j}{\Gamma(j)\cdot j!}[\psi(j+\nu+1) - \psi(\nu+1-j)]\cdot\left(\sin\frac{\theta}{2}\right)^{2j}$$

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Анотація. *У роботі розглядається задача кручення усіченого конуса з внутрішньою сферичною тріщиною. Метою дослідження є визначення напружено-деформованого стану конуса за наявності тріщини і прикладеного вісесиметричного дотичного навантаження на поверхню. Для вирішення задачі використано інтегральне перетворення Лежандра, яке дозволяє звести вихідну крайову задачу до одномірної розривної крайової задачі. Рішення представлено у вигляді суми неперервних і розривних складових, отриманих із застосуванням функції Гріна. Невідомий стрибок переміщення на тріщині визначається з інтегрального рівняння, яке вирішено методом ортогональних многочленів із використанням многочленів Чебишева. У результаті обчислено коефіцієнт інтенсивності напружень поблизу тріщини та проведено чисельний аналіз впливу геометричних параметрів на напружений стан. Отримані результати є важливими для оцінки міцності та стійкості конструкцій із тріщинами та можуть бути використані в інженерній практиці.*

Ключові слова: *зрізаний конус, сферична тріщина, коефіцієнт інтенсивності напружень, інтегральне перетворення, метод ортогональних многочленів.*

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