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RESONANT AND NON-RESONANT ELECTROMAGNETIC FIELDS IN QUASI-OPTICAL OPEN RESONATORS

Konstantin Lukin
Dr. of Sciences, Prof., IEEE Fellow
ORCID: 0000-0001-9998-9207
Institute for Radiophysics & Electronics NASU, Kharkiv, 12 Ac. Proskura St., 61085
and Instituto Tecnológico de Aeronáutica (ITA), Brasil
Lidiya Yurchenko
PhD, Senior Researcher
ORCID: 0009-0008-2608-7761
Institute for Radiophysics & Electronics NASU, Kharkiv, 12 Ac. Proskura St., 61085

Abstract. *A new approach to the solving the problems of electromagnetic fields excitation is suggested for quasi-optical open resonators (OR) described in terms of Fredholm's integral equations. The general structure and character of the dependence of the non-resonant part of the excited field in the vicinity of a single quasi-eigen (resonant) mode on the excitation frequency and Q-factor of this mode are defined. A formal transition to the equations of the well-known theory of quasi-optical OR excitation is shown. A new form of the power balance equations for active and reactive powers in the system "OR + Source" is proposed. These equations are suitable for analyzing the non-resonant radiation and calculating the conversion coefficient of the source radiating energy into the energy of the OR quasi-eigen mode. The suggested approach is applicable in design of OR based microwave, terahertz and photonics devices.*

Keywords: *open resonator (OR), quasi-optical OR, OR quasi-eigen mode, resonant and non-resonant fields, field energy balance*

Introduction

Open Resonators (OR) are used in design of optics, infrared, terahertz (THz), microwave and photonic devices [1,2]. One of the main challenges of the OR based devices investigations lays in the elaboration of the OR excitation theory. Excitation theory of a cavity essentially uses representation of the EM fields as a superposition of the cavity's normal modes with a coefficients which are solutions of the related spectral theory and forms an orthogonal basis in functional space. However, there is no normal modes in ORs which could form a complete system of eigenfunctions of the discrete spectrum, and, consequently, there is no representations of the excited field in the form of the superposition of these modes with coefficients calculated via the source current. From the physical point of view, this is due to the fact that in addition to the diffraction energy losses of the OR quasi-eigen modes, in the ORs there is always take place a non-resonant radiation of the source, which is not associated with the diffraction energy loss of the OR quasi-eigen mode. In [3-5] a new approach is developed in the excitation theory for two-dimensional ORs by internal sources. The proposed approach uses efficient algorithms of the mathematical theory of electromagnetic waves diffraction which is based on the method of Riemann–Hilbert Problem [6]. This method enabled adequate formulation of the OR spectral problem [7-8] and was used for constructing the Green's function for the considered class of ORs [3-5]. The found solutions of the OR excitation problem enabled calculating the fields both inside and outside the OR resonant volume with an arbitrary relationship between the wavelength and the OR dimensions.

In the present paper, a similar approach is elaborated within the framework of approximate methods of diffraction theory, which allow calculating fields in quasi-optical OR having complex-profile mirrors, in particular in the OR of a Diffraction Radiation Generator (DRG) containing a diffraction grating [1,2].

Resonant and Non-Resonant Fields Excited by Arbitrary Sources in Quasi-Optical Open Resonators

Approximate methods of diffraction theory of electromagnetic waves enable calculating fields in quasi-optical OR having dimensions of its mirrors essentially exceeding the field wavelength. The required general relations for the problems of excitation of quasi-optical ORs in the Kirchhoff approximation were obtained long time ago with the corresponding mathematical justification. [9,10]. In this section, we show that representations similar to those in [3-5] are also valid for *quasi-optical* OR, and the entire scheme for calculating the fields in ORs excited by linear/nonlinear currents is applicable within the framework of a step-by-step approach to self-consisting solving the OR excitation problems.

Let us consider a quasi-optical resonator with perfectly conducting mirrors and dimensions that ensure the fulfillment of the conditions $kR \gg 1$ and $\lambda \ll a$, where R is the smallest distance between the mirrors and a is the aperture of the mirrors. In this approximation, the integral equation for the H_z component of the field on the OR mirror has the form [10]:

$$H_z(S_1) = \left(\frac{ik}{2\pi}\right)^2 \int \frac{e^{ikR_{12}}}{R_{12}} \int \frac{e^{ikR_{21}}}{R_{21}} H_z(S_1^*) ds_1 ds_2 - \frac{ik}{2\pi} \int \frac{e^{ikR_{12}}}{R_{12}} F_2(S_2) ds_2 + F_1(S_1), \quad (1)$$

where $R_{12} = |S_1 S_2|$ is the distance between integration points over the surface S_1, of the first and second mirror of open OR; $R_{21} = |S_2 S_1|$ is the distance between integration points over the surface S_2 of the second and the first mirrors;

$$F_i(S_i) \overset{d}{=} \int_{V_0} \frac{e^{ikR_i}}{R_i} f(\theta) dv \quad (2)$$

$f(\theta) = i\left(\frac{4\pi}{kc}\right) j(\theta)$ is the source function; $R_i = |S_i \theta|$ is the distance between integration points over the surface S_i, and the source, θ. Integration in Eq.(1) is carried out over the surfaces of the OR mirrors, and in Eq.(2) integration is carried out over the volume V_0 occupied by the source.

The integral equation (1) is obtained in the Kirchhoff's approximation, which does not take into account the current leakage to the shadow side of the mirrors. However, when solving the problem, the total field is not divided into resonant and non-resonant parts, therefore, the calculation of the field excited in the OR by the source using this equation is performed with accounting the non-resonant radiation. Eq.(1) can be solved by the iteration method in the same way as Fox and Lee did in [9]. In this case, it is also convenient to use the iteration solution procedure, which corresponds to the above-described step-by-step approach to solving the problem of excitation of resonators by arbitrary sources. First, the field distribution (or equivalent surface currents) on one of the mirrors and the initial value of the source current (for example, found in the approximation of a given field or in a linear approximation) are specified. Then, according to Eq.(1), we find the distribution of the field that arises on the surface of the same mirror after one reflection from the second mirror. After that calculate the distribution of the source current and find the field again, etc. Thus, Eq.(1) in principle allows solving the problems of non-relativistic Diffraction Electronics [1] with an unfixed field structure taking into account non-resonant radiation.

From Fredholm's theory it is known that the solution of Eq. (1) can be represented through the resolvent kernel $\Gamma(S_1, S_1^*, \lambda(k))$:

$$H_z(S_1) = F(S_1) + \lambda(k) \int \Gamma(S_1, S_1^*, \lambda(k)) F(S_1^*) ds_1 \quad (3)$$

where

$$F(S_1) \overset{d}{=} -\frac{ik}{2\pi} \int F_2(S_2) \frac{e^{ikR_{12}}}{R_{12}} ds_2 + F_1(S_1)$$

and the resolvent kernel is:

$$\Gamma(S_1, S_1^*, \lambda(k)) = \lambda(k)\frac{\widetilde{D}(S_1, S_1^*, \lambda(k))}{D(\lambda)},$$

where $D(\lambda)$ and $\widetilde{D}(S_1, S_1^*, \lambda(k))$ are Fredholm determinant and Minor of the Fredholm determinant, respectively [10].

The zeros of the Fredholm determinant lie in the lower half-plane of the complex parameter and coincide with the eigenvalues of the homogeneous Fredholm equation. Thus, as in the case of rigorous methods of diffraction theory, the solution of the excitation problem is represented as a meromorphic function Eq.(3) of the spectral parameter $\lambda(k)$, the poles of which coincide with the spectrum of the OR *eigen-frequencies*.

This means that, applying the Cauchy's residue theorem to the function $H_z(\xi)/(k-\xi)$ we obtain expressions for the excited fields in the form of a sum of resonant terms and a non-resonant term:

$$H_z^{sc}(S_1) \stackrel{d}{=} H_z(S_1) - F(S_1) = \sum_{s=1}^{M} \frac{C_s(k)H_s(k_s)}{k-k_s} + \frac{1}{2\pi i}\oint_{C_s} \frac{\phi(\xi, S_1)}{\xi - k} d\xi, \tag{4}$$

where

$$\phi(\xi, S_1) \stackrel{d}{=} \lambda(\xi)\int_{S_1} \Gamma_1(\xi, S_1, S_1^*) F_1(S_1^*) ds_1,$$

$$\frac{1}{2\pi i}\oint \phi(\xi, S_1) d\xi \stackrel{d}{=} C_s(k) H_s(k_s)$$

is the residue of the function $\phi(\xi)$ at the *s-th* pole of the resolvent $\Gamma(S_1, S_1^*, \lambda(k))$.

The representation of the solution for equation (1) in the form (4) indicates the structure of the excited field and allows us to formulate an algorithm for calculating the power of non-resonant radiation. Using the iteration procedure, we solve equation (1) and find the total field excited in the OR, which allows us to calculate the total radiation losses in the system: *OR + source*. Then, for the same resonator, the power of diffraction losses is determined by solving the problem using the formulas of the theory [9-10], which corresponds to the using only one resonant term in Eq.(4). As a result of the comparison, we find the power of non-resonant radiation. Note that in this case, we can use the expressions for the quasi-eigenfunctions obtained by the parabolic equation method, since in the integral equation (1), in fact, the Green's function of the parabolic equation is used.

Energy Balance Equations for OR with Source

The exact balance equations for OR are a direct consequence of the complex power theorem. In order to write it in a form convenient for the analysis of resonant systems, we introduce into consideration energy W_E of electric and energy W_H magnetic fields in the resonant volume V_R:

$$W_E = (8\pi)^{-1} \int_{V_R} |\bar{E}|^2 d\,v \quad \text{and} \quad W_H = (8\pi)^{-1} \int_{V_R} \left|\vec{H}\right|^2 dv \tag{5a}$$

and their sum and difference:

$$W = W_E + W_H \quad \text{and} \quad \Delta W = W_E - W_H. \tag{5b}$$

By the resonant volume we mean a volume limited by a certain surface that covers the OR and the source. In open resonators, the resonant volume should be chosen as a volume limited by mirrors and caustic surfaces. We will limit ourselves to the analysis of monochromatic fields with time dependence $e^{-i\omega t}$. Quasi-eigen modes of the OR are characterized by complex eigen-frequencies

$$\omega_s = \omega_s{}' - i\omega_s{}'', \ (\omega_s{}', \omega_s{}'' > 0).$$

Taking this circumstance into account when deriving the complex power theorem from Maxwell's equations, we obtain the balance equations for the eigen-modes in the following form:

$$\omega_s{}' = \frac{1}{2}\frac{\sum{}''(\omega_s)}{\Delta W(\omega_s)}; \quad \omega_s{}'' = \frac{1}{2}\frac{\sum{}'(\omega_s)}{W(\omega_s)} \tag{6}$$

where $\sum = \sum{}' + i\sum{}'' = \frac{c}{4\pi}\int_S [\vec{E}, \vec{H}] d\vec{s}$ is complex energy flux (radiant power) through a surface S, limiting the resonant volume V; $\vec{E}$ and $\vec{H}$ are electric and magnetic fields, respectively; c is the speed of light.

Since for a fixed geometry of an OR its quasi-eigen frequency $\omega_s = const$, then from Eq.(6) it follows that the ratio of the real (imaginary) part of the radiation power from the volume V to the sum (difference) of the energies of the electric and magnetic fields of the quasi-eigen modes does not depend on the choice of surface S. Let us recall that ω_s' determines the frequency of the quasi-eigen mode oscillations, and ω_s'' is the decrement of quasi-eigen mode attenuation with time due to the radiation of the field to infinity. From Eq.(6) it is evident that the rate of change of the instantaneous phase of quasi-eigen mode (eigen-frequency ω_s'') is determined by the ratio of their reactive characteristics ($\sum''$ and ΔW), while the attenuation rate of the amplitude of these oscillations is determined by the ratio of their active characteristics ($\sum'$ и W).

Introducing into consideration the quality factor of quasi-eigen mode according to the well-known formula $Q(\omega_s) = \omega_s'/2\omega_s''$, from the second formula (6) we obtain

$$Q(\omega_s) = \frac{\omega_s' W(\omega_s)}{\sum'(\omega_s)} \qquad (7)$$

Eq.(7) agrees well with the classical definition of the Quality factor of an oscillatory circuit or cavity, since (7) represents the ratio of the electromagnetic energy in the volume V averaged over the period of oscillation to the energy loss, which in this case is determined by the average radiation power from this volume.

Let us now consider the balance equations for the case of forced oscillations excited in the OR by a source (with frequency ω) located inside the resonant volume. The balance equations for active components follow from the complex power theorem:

$$\sum'(\omega) = P_a(\omega) \qquad (8)$$

and reactive:

$$2\omega\Delta W(\omega) + \sum''(\omega) = P_r(\omega) \qquad (9)$$

power. The following notations are introduced above: $P(\omega) = P_a(\omega) + i\,P_r(\omega)$ is complex power of interaction of the source with the OR field; V_e - the volume occupied by a source with a current density $\vec{j}$. The function $2\omega\Delta W(\omega)$ characterizes the reactive power of the excited OR mode.

Let's introduce, by analogy with (7), the quality factor of forced oscillations:

$$Q(\omega) = \frac{\omega W(\omega)}{\sum'(\omega)}. \qquad (10)$$

Using Eq.(7) and Eq.(10), we may reformulate Eq.(8) and Eq. (9) as follows:

$$2\omega_s''W(\omega) = \sigma(\omega, \omega_s)P_a(\omega), \tag{11}$$

$$2(\omega - {\omega_s}'\delta(\omega, \omega_s))\Delta W(\omega) = P_r(\omega), \tag{12}$$

where

$$\sigma(\omega, \omega_s) \equiv \frac{{\omega_s}'Q(\omega)}{\omega Q(\omega_s)}, \quad \delta(\omega, \omega_s) \equiv -\frac{1}{2}\frac{\Sigma''(\omega)}{\omega_s' \Delta W(\omega)} \tag{13}$$

With ideal conductivity of metal surfaces, the left-hand side of Eq.(11) can be interpreted as diffraction losses of the field energy contained in the resonant volume at the excitation frequency. Then the value on the right-hand side of Eq.(11) can be considered as that part of the source power which is necessary to compensate for the diffraction losses of the resonant oscillation. The remaining power is radiated directly into free space or is scattered non-resonantly by the OR mirrors. Thus, the total radiation power of the source from the OR can be conditionally divided into two qualitatively different parts. One of them is due multiple re-reflections of the field by the OR mirrors, it is associated with diffraction losses and has a resonant nature. The second part is caused by a single scattering on OR mirrors and direct radiation of the source into free space and forms non-resonant radiation. When sweeping the frequency of the source in the receiving device located outside the resonant volume, the recorded signal will have the form of a narrow peak (the width of which characterizes the quality factor of forced oscillations) on a flat pedestal caused by non-resonant radiation.

The Q-factor (10) of the forced oscillations introduced above, and therefore the value of $\sigma(\omega, \omega_s)$, depend not only on the geometry of the OR and the frequency of the source, but also on its spatial structure, location and interaction with the OR field. Therefore, the value $\sigma(\omega, \omega_s)$ or $Q(\omega)$ can be used as a criterion for the efficiency of converting the source energy into the energy of the OR resonant mode. Another conclusion that follows from these results is that in the energy balance analysis of oscillations excited in an OR with a source located inside the resonant volume, the OR and the source should be considered as a single system.

In the case of high Q-factor oscillations, characterized by the conditions $\omega \approx {\omega_s}'$ and $Q(\omega_s) \gg 1$, the source has little effect on the field structure in the OR. If the power radiating out of the OR resonant volume is caused only by diffraction losses, then $Q(\omega) \approx Q(\omega_s)$ (i.e. $\sigma \approx 1$) and $\sum''(\omega) \approx 0$. Moreover, in this case $\omega^2 W_H \approx \omega_s^2 W_E$. Taking these relationships into account, Eq.(11) and Eq.(12) may be reduced to the known equations for the balance of active and reactive powers for cavities (closed resonators), which have the form:

$$2\omega_s'' W(\omega) = P_a(\omega) \text{ and } 2(\omega - \omega_s{}')W(\omega) = P_r(\omega) \qquad (14)$$

Eq.(14) are also used to describe field excitation in OR with high Q-factor.

It should be noted that since for the forced oscillations the energy flow $\sum'$ does not depend on the choice of the surface limiting the volume V_R (see Eq.(5)), then, by increasing this volume, the value of Eq.(10) can be made arbitrarily large. However, it should be borne in mind that the expression on the left-side of Eq. (11) has the meaning of the power of diffraction losses only if the resonant volume V_R is chosen as the integration volume in (5). Therefore, the comparison of the Q-factors of forced and free oscillations should be carried out only for the resonant volume V_R.

Conclusions

On the basis of the spectral theory of a two-dimensional OR with ideally conducting cylindrical mirrors [7,8], an expression for the Green's function of such OR was derived in [3-5]. A similar approach to the solution of excitation problems is also developed for *quasi-optical* OR described in terms of Fredholm’s integral equations. The general structure and character of the dependence of the resonant and non-resonant parts of the field of OR excited in the vicinity of a single OR resonant (*quasi-eigen*) mode are obtained as a functions of the source frequency and Q-factor of this mode. A new form of balance equations of active and reactive powers in the system "OR + Source" is proposed, which is convenient for analyzing non-resonant radiation. A formal transition to the balance equations of the traditional theory of excitation of OR quasi-eigen mode having a high Q-factor is shown. A new method is proposed for calculating the coefficient of conversion of the radiation energy of a given source into the energy of the resonant (*quasi-eigen*) mode of the OR, which consist in calculating the ratio of the Q-factor of the forced oscillations to the diffraction Q-factor of the OR *quasi-eigen* mode. In the same approximation, equations were formulated and corresponding algorithms for calculating EM fields in the OR with diffraction grating were elaborated, which are suitable for describing self-oscillatory regimes in Diffraction Electronics [1] devices with a non-fixed field structure and which account *non-resonant* radiation. The suggested theory may be applied for design of active and passive elements of microwave, terahertz and photonic systems based on quasi-optical ORs.

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